Phys 410
Fall 2014
Lecture #23 Summary
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We considered the most general coupled oscillator problem – N particles in three dimensions coupled to each other by means of springs or any other types of forces that produce a stable equilibrium configuration. This system has n generalized (perhaps normal) coordinates, where in general $n \leq 3N$. The generalized coordinates are written as $\vec{q} = (q_1, q_2, \ldots q_n)$. We assume that only conservative forces act between the particles, hence (as known from previous studies) the potential energy is a function only of the coordinates: $U = U(\vec{q})$. The kinetic energy is that of all of the particles in the system: $T = \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha \dot{\vec{r}}_\alpha^2$. The "raw" coordinates $\vec{r}_\alpha$can be written in terms of the generalized coordinates as $\vec{r}_\alpha = \vec{r}_\alpha(q_1, q_2, \ldots q_n)$, where it is assumed that no explicit time-dependence is required to write down this transformation. The kinetic energy can be written as $T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\,\dot{q}_i\dot{q}_j$, where the matrix $\bar{\bar{A}}$ is defined as $A_{ij} \equiv \sum_{\alpha=1}^{N} m_\alpha \frac{\partial \vec{r}_\alpha}{\partial q_i} \cdot \frac{\partial \vec{r}_\alpha}{\partial q_j}$. Note that the double pendulum kinetic energy (see the Lagrangian in the last lecture) has a kinetic energy of this form, including a $\dot{q}_1\dot{q}_2$ term. Note that the matrix $\bar{\bar{A}}$ is a function of the generalized coordinates as well: $\bar{\bar{A}} = \bar{\bar{A}}(\vec{q})$. We now have the full Lagrangian of this generalized coupled oscillator problem $\mathcal{L} = T(\vec{q}, \dot{\vec{q}}) - U(\vec{q})$.

To make further progress we next considered the small oscillation motion of the system around a stable equilibrium point. This means that we will keep terms only up to second order in the variables. By a shift of the origin, we can make the stable equilibrium point appear at the point $\vec{q} = (0, 0, \ldots 0)$. We then did a Taylor series expansion of the potential around this point and kept terms up to second order, yielding $U(\vec{q}) = \frac{1}{2}\sum_{i,j} K_{ij} q_i q_j$, where the matrix elements of $\bar{\bar{K}}$ are the curvatures of the potential with respect to the generalized coordinates: $K_{ij} \equiv \left.\frac{\partial^2 U}{\partial q_i \partial q_j}\right|_{\vec{q}=0}$. The kinetic energy is already quadratic in the variables, so we simply evaluate it at $\vec{q} = 0$ to yield $T = \frac{1}{2}\sum_{i,j} A_{ij}(0)\dot{q}_i\dot{q}_j \quad = \frac{1}{2}\sum_{i,j} M_{ij}\dot{q}_i\dot{q}_j$, where the mass matrix $\bar{\bar{M}}$ is the $\bar{\bar{A}}$ matrix evaluated at the equilibrium position $\vec{q} = (0, 0, \ldots 0)$. The Lagrangian $\mathcal{L} = T(\dot{\vec{q}}) - U(\vec{q})$ is now a homogeneous quadratic function of the coordinates and their time-derivatives, and the matrices $\bar{\bar{M}}$ and $\bar{\bar{K}}$ are constant symmetric real matrices.

There are n Lagrange equations to set up and solve. We wrote down the equations and found that the set of n equations are summarized beautifully in a simple matrix equation: $-\bar{\bar{K}}\vec{q} = \bar{\bar{M}}\ddot{\vec{q}}$. We can solve this equation using the same method employed before, just generalized to n coordinates. We use the standard complex *ansatz* for the solution vector: $\vec{q}(t) = Re[\vec{C}e^{i\omega t}]$, where $\vec{C} = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix}$, and the C_i are complex constants. This assumes that all of the coordinates adopt the same oscillation frequency ω, and each oscillator will adopt its own amplitude and phase (through the choice of $C_i = A_i e^{-i\delta_i}$). Putting this into the matrix equation yields $(\bar{\bar{K}} - \omega^2\bar{\bar{M}})\vec{C} = 0$. To get a non-trivial solution for $\vec{C}$, we demand that $det(\bar{\bar{K}} - \omega^2\bar{\bar{M}}) = 0$. This yields an n-th order equation for ω^2, with n real solutions (we know this because the matrix $\bar{\bar{K}} - \omega^2\bar{\bar{M}}$ is real and symmetric). The n normal modes follow by standard linear algebra. The most general solution is a linear combination of motion in all of the normal modes, each with distinct amplitude and phase. The motion in a given normal mode may involve a coordinated motion of all the particles in the system! The Kuramoto model describes a collection of coupled oscillators, each with a unique natural frequency and coupled to all other oscillators through a nonlinear coupling. It describes the synchronization (or lack thereof) of generators making up the power grid, as well as the coordinated oscillations of fireflies in Tennessee, among other things.

We then turned to a discussion of Special Relativity. We began by reviewing the Galilean transformation between inertial reference frames, and showed that Newton's second law of motion holds in the same form in all inertial reference frames. This result relies on the Galilean velocity addition formula between reference frames. However, it was discovered that Galilean invariance does not apply to Maxwell's equations (which are actually Lorentz invariant) by examining the measurement of the speed of light in a moving reference frame. The Michelson-Morley experiment showed that the measured speed of light is the same in all directions for all inertial observers. Hence there must be something more going on than simple Galilean transformations between reference frames.

Einstein made two postulates:

1) If S is an inertial reference frame and if a second frame S' moves with constant velocity relative to S, then S' is also an inertial reference frame.
2) The speed of light (in vacuum) has the same value *c* in every direction in all inertial reference frames.

The first postulate points out that there is no "special" reference frame which is absolutely at rest and somehow 'better' than any other reference frame. It also implies that all the laws of physics (including Maxwell's equations) should take on the same form in all inertial reference frames. Again it says that there is no single inertial reference frame in which the laws of physics are simpler, or have fewer terms, than any other reference frame. The trick will be finding how to transform all of the coordinates from one inertial reference frame to another to preserve the form of the laws of physics. The second postulate codifies the results of the Michelson-Morley experiment, and leads to many non-intuitive results.

We examined the relativity of time by considering two reference frames, one with railroad tracks at rest (S), and the other (S') on a train moving down the tracks at a high rate of speed (V). Consider a light-clock on the train (frame S') that sends a brief flash of light from the floor to the ceiling, where it bounces off of a mirror, and then back to a detector that is co-located with the source on the floor. The time interval for the round trip of the light beam is $\Delta t' = 2h/c$, where h is the height of the train and c is the speed of light, as measured in S'. An observer (or really a set of observers) in S see the light follow a triangular trajectory as the train wizzes by. From the geometry of the experiment, and the second postulate, those observers attribute a time interval for the "round trip" of $\Delta t = \gamma \Delta t'$, where $\gamma = 1/\sqrt{1-\beta^2}$, and $\beta = V/c$. Since $\gamma > 1$ the two observers do not agree on how much time elapsed on the light-clock! This shows that the Galilean idea of universal time for all inertial observers is incorrect. In addition, because γ diverges as $V \to c$, it says that there is a speed limit for inertial reference frames: $V < c$. (This also means that we cannot address the question of what the world looks like from the reference frame of a photon travelling at the speed of light, at least with this formalism.)

The first postulate implies the equality of all inertial reference frames, so why is the result $\Delta t = \gamma \Delta t'$ asymmetric between the two inertial reference frames? The difference arises because the time interval was measured at a single fixed location in S' while it was measured at two distinct locations in S. The measurement of a time interval at a fixed location in an inertial reference frame is called the 'proper time interval' and is denoted Δt_0. Measurements of these two events taken from any other inertial reference frame moving with respect to this one will result in a dilated time interval measurement $\Delta t = \gamma \Delta t_0$.